*Subgroups of Order a Power of p in the General and Special m-ary Linear Homogeneous Groups in the $GF[p^n]$.**

By Leonard Eugene Dickson.†

1. It would seem that the most effective method of determining all the subgroups of order a multiple of p of a linear group in the Galois field of order p^n is that based upon a complete knowledge of the subgroups of order a power of p. This method has proved successful for the ternary groups‡ and, as I will show on another occasion, also for the quaternary groups.

The present investigation proceeds far enough to give a clear insight into the nature of the simple laws pervading the subject. It is hoped that the results are capable of extension by induction to all powers of p. To indicate the difficulty of this step, it may be remarked that its completion would give the means of deriving at once an explicit list of all groups of order a power of a prime, and simultaneously all the subgroups of each.

A second aim of the paper was to furnish data for the problem of the determination of all m-ary groups for low values of m.

Following Lie, I write (A, B) for the commutator $A^{-1}B^{-1}AB$. I employ the usual notation $B_{ij\alpha}$ for the transformation which alters only ξ_i, replacing it by $\xi_i + \alpha\xi_j$. We have the simple relation§

$$(B_{it\alpha}, B_{tj\beta}) = B_{ij-\alpha\beta}, \qquad (i > t > j), \tag{1}$$

by use of which computations are reduced to a minimum.

* Presented before the American Mathematical Society at St. Louis, Sept. 16, 1904. Some special cases treated by other methods have been given by the writer, Bull. A. M. S., Vol. 10 (1904), pp. 385–397; Quarterly Jour., Vol. 36 (1905), pp. 373-384.

† Research assistant to the Carnegie Institution of Washington.

‡ American Journal of Mathematics, Vol. 27 (1905), pp. 189-202.

§ It follows from it that $SLH(m, pn)$, $m > 2$, is a perfect group.

2. Let G be either the general or special linear homogeneous group, $GLH(m, p^n)$ or $SLH(m, p^n)$. The highest power of p dividing the order of G is $p^{\mu n}$, $\mu = \frac{1}{2} m (m-1)$. The totality of the transformations in the $GF[p^n]$

$$[\alpha_{ij}]: \quad (\alpha_{ij}) \text{ with every } \alpha_{ij} = 0 \ (j > i), \quad \alpha_{ii} = 1 \tag{2}$$

forms a subgroup $G_{p^{\mu n}}$ The law of composition is

$$\left.\begin{array}{l} [\alpha][\beta] = [\gamma], \\ \gamma_{ii-1} = \beta_{ii-1} + \alpha_{ii-1}, \quad \gamma_{ij} = \beta_{ij} + \alpha_{ij} + \sum_{k=j+1}^{i-1} \beta_{ik}\alpha_{kj}, \qquad (j < i-1). \end{array}\right\} \tag{3}$$

Since every subgroup of order a power of p of G is conjugate with a subgroup of $G_{p^{\mu n}}$, it suffices to determine those of $G_{p^{\mu n}}$.

In case a subgroup of $G_{p^{\mu n}}$ can be defined by certain relations $r_1 = 0, \ldots, r_s = 0$ between the α_{ij}, we denote it $\{r_1 = 0, \ldots, r_s = 0\}$.

3. Let the $GF[p^n]$ be defined by an irreducible congruence

$$\rho^n \equiv \sum_{i=0}^{n-1} r_i \rho^i, \qquad (\text{mod } p). \tag{4}$$

Set

$$\alpha_{ij} = \sum_{k=0}^{n-1} a_{ijk}\rho^k, \quad \beta_{ij} = \sum_{k=0}^{n-1} b_{ijk}\rho^k, \quad \gamma_{ij} = \sum_{k=0}^{n-1} c_{ijk}\rho_k. \tag{5}$$

Relations (3) in the $GF[p^n]$ are equivalent to a set of congruences (3′) mod p, including

$$c_{ii-1k} \equiv b_{ii-1k} + a_{ii-1k}, \qquad (i = 2, \ldots, m; \ k = 0, \ldots, n-1). \tag{6}$$

If, for every pair of operators $[\alpha]$ and $[\beta]$ of a subgroup H of $G_{p^{\mu n}}$, $\phi(c) \equiv \phi(b) + \phi(a)$, we say that H is *additive with respect to* $\phi(a)$. Thus $G_{p^{\mu n}}$ is additive with respect to every a_{ii-1k}.

If H is additive with respect to a_{ijk}, then $[\alpha]^p = [\alpha']$ has $a'_{ijk} \equiv 0$, and $([\alpha], [\beta]) = [\delta]$ has $d_{ijk} \equiv 0$. In particular, all the commutators of $G_{p^{\mu n}}$ have every $d_{ii-1k} \equiv 0$.

4. LEMMA. *If the pth power of every operator of a group G_{p^g} belongs to its commutator subgroup G_{p^c}, there are exactly $(p^{g-c} - 1)/(p-1)$ subgroups of order p^{g-1} in G_{p^g}.*

Indeed, G_{p^g}/G_{p^c} is abelian of type $(1, 1, \ldots, 1)_{g-c}$ and contains every H/G_{p^c}, H being a subgroup of order p^{g-1}.

5. The commutator subgroup K of $G_{p^{\mu n}}$ is formed of the operators $[\delta_{ij}]$ with every $\delta_{ii-1} = 0$ (§3) and every δ_{ij} $(j < i - 1)$ arbitrary in the $GF[p^n]$. Indeed, in view of (1), K contains every $B_{ij\delta}$ $(j < i - 1)$ since each $B_{ij\alpha}$ $(j < i)$ belongs to $G_{p^{\mu n}}$. Then $[\alpha]^p$ belongs to K. Hence follows the

THEOREM. *The* $(p^{(m-1)n} - 1)/(p - 1)$ *subgroups of order* $p^{\mu n - 1}$ *of* $G_{p^{\mu n}}$ *are defined by a linear homogeneous relation between* a_{ii-1k} $(i = 2, \ldots, m;$ $k = 0, \ldots, n - 1)$.

For $n = 1$, we give to the subgroups the notation

$$\left\{ \sum_{i=2}^{m} c_i a_{ii-1} = 0 \right\}, \text{ not every } c_i \text{ zero.} \tag{7}$$

6. THEOREM.—*Let H be a subgroup defined by r independent linear homogeneous relations between the a_{ss-1k} $(k = 0, 1, \ldots, n - 1)$. If $r < n$, the $(p^{(m-1)n-r} - 1)/(p - 1)$ subgroups of index p under H are given by annexing any linear homogeneous relation between the a_{ii-1k} $(i = 2, \ldots, m;$ $k = 0, \ldots, n - 1)$ independent of the r relations. If $r = n$, so that $H = \{a_{ss-1} = 0\}$, and $2 < s < m$, the $(p^{mn} - 1)/(p - 1)$ subgroups of index p under H are given by annexing any linear homogeneous relation between the a_{ii-1k} $(i \neq s)$, a_{ss-2k}, $a_{s+1s-1k}$; while for $s = 2$ or $s = m$, the $(p^{(m-1)n} - 1)/(p - 1)$ subgroups are obtained from the preceding by suppressing the a_{ss-2k} or $a_{s+1s-1k}$, respectively.*

H contains every $B_{ij\alpha}$ $(j < i)$ except certain $B_{ss-1\alpha}$. By (1),

$$(B_{ii-1\alpha}, B_{i-1i-2\beta}) = B_{ii-2\gamma},\ (B_{ii-2\alpha}, B_{i-2j\beta}) = B_{ij\gamma}\ (i - j > 3), \tag{8}$$

$$(B_{ii-1\alpha}, B_{i-1i-3\beta}) = B_{ii-3\gamma},\ (B_{ii-2\alpha}, B_{i-2i-3\beta}) = B_{ii-3\gamma}, \tag{9}$$

where $\gamma = -\alpha\beta$. By (8) the commutator group K of H contains every $B_{ii-2\gamma}$ $(i \neq s, s + 1)$ and every $B_{ij\gamma}$ $(i - j > 3)$. Applying $(9)_1$ if $i \neq s$ and $(9)_2$ if $i = s$, we find that K contains every $B_{ii-3\gamma}$.

If $r < n$, H contains a $B_{ss-1\alpha}$, $\alpha \neq 0$. By $(8)_1$ for $i = s$ and $i = s + 1$, K contains $B_{ss-2\gamma}$ and $B_{s+1s-1\gamma}$, where $\gamma = -\alpha\beta$ is arbitrary since β is. Hence K contains every $B_{ij\delta}$ $(j < i - 1)$.

If $r = n$, so that the relations require $\alpha_{ss-1} = 0$, H is additive with respect to α_{ss-2} and α_{s+1s-1} by (3). Hence K is composed of the operators $[\delta_{ij}]$ with every $\delta_{ii-1} = 0, \delta_{ss-2} = 0$, $\delta_{s+1s-1} = 0$, and the remaining δ_{ij} $(j < i)$ all arbitrary.

7. LEMMA. *If H is a subgroup defined by certain relations R between the a_{ii-1k} $(i = s, \ldots, s+t;\ k = 0, \ldots, n-1)$ not requiring that $\alpha_{ss-1} = 0$ if $s > 2$, nor that $\alpha_{s+ts+t-1} = 0$ if $s + t < m$, nor that $\alpha_{s+2s+1} = 0$ if $t > 1$, the commutator subgroup K of H contains every $B_{ij\alpha}$ $(j < i-1)$ except possibly $B_{ii-2\alpha}$ $(i = s+1, \ldots, s+t)$.*

H contains every $B_{ij\alpha}$ $(j < i)$ except $B_{ww-1\alpha}$ $(w = s, \ldots, s+t)$; also,

$$B_\alpha:\quad \xi'_i = \xi_i + \alpha_{ii-1}\xi_{i-1}, \qquad (i = s, \ldots, s+t), \tag{10}$$

the α_{ii-1} being subject to relations R. The inverse of B_α is

$$B_\alpha^{-1}:\quad \xi'_i = \xi_i + \sum_{j=1}^{i-s+1} (-1)^j \alpha_{ii-1}\alpha_{i-1i-2}\cdots\alpha_{i-j+1i-j}\xi_{ij}\,. \qquad (i = s, \ldots, s+t). \tag{11}$$

In view of (8) and (9), K contains every

$$B_{ij\gamma}(i - j > 3),\ B_{ii-2\gamma}(i \neq s, \ldots, s+t+1),\ B_{ii-3\gamma}(i \neq s+2, \ldots, s+t). \tag{12}$$

If $s > 2$, $(B_{s-1s-2\tau}, B_\alpha) = B_{ss-2\sigma}$, where $\sigma = \tau\alpha_{ss-1}$ may be made arbitrary by choice of τ. If $s + t < m$,

$$(B_{s+t+1s+t\tau}, B_\alpha):\ \xi'_{s+t+1} = \xi_{s+t+1} + \tau\sum_{j=1}^{t+1}(-1)^j\alpha_{s+ts+t-1}\cdots\alpha_{s+t-j+1s+t-j}\xi_{s+t-j}\,.$$

Multiplying this on the right by $\prod_{j=2}^{t+1} B_{s+t+1s+t-j\lambda_j}$, which belongs to K, we obtain $B_{s+t+1s+t-1\lambda}$ by suitably choosing the λ_j; here $\lambda = -\tau\alpha_{s+ts+t-1}$ may be made arbitrary. If $t > 1$ and $s + 1 \leqq l \leqq s + t - 1$,

$$C_l \equiv (B_{ll-2\tau}, B_\alpha):\ \xi'_l = \xi_l + \tau f,\ \xi'_{l+1} = \xi_{l+1} + \tau\alpha_{l+1l}\,\xi_{l-2} + \tau\alpha_{l+1l}f,$$

$$f \equiv \sum_{j=1}^{l-s-1} (-1)^j \alpha_{l-2l-3}\cdots\alpha_{l-j-1l-j-2}\xi_{l-j-2}\,.$$

In particular, $C_{s+1} = B_{s+2s-1\kappa}$, where $\kappa = \tau\alpha_{s+2s+1}$ may be made arbitrary. Then by choice of λ_1 and λ_2,

$$C_{s+2} B_{s+2s-1\lambda_1} B_{s+3s-1\lambda_2} = B_{s+3s\lambda},$$

where λ is arbitrary. Next, by choice of μ_1, μ_2, ν_1, ν_2,

$$C_{s+3} B_{s+3s\mu_1} B_{s+3s-1\mu_2} B_{s+4s\nu_1} B_{s+4s-1\nu_2} = B_{s+4s+1\nu},$$

where ν is arbitrary. It follows in this way that K contains every $B_{ii-3\gamma}$ $(i = s+2, \ldots, s+t)$. But these with $B_{ss-2\sigma}$, $B_{s+t+1s+t-1\lambda}$ and (12), give all stated in the lemma.

8. THEOREM.—*Let K be the commutator subgroup of*

$$\left\{ \sum_{i=s}^{s+t} c_i \alpha_{ii-1} = 0 \right\}, c_s \neq 0, \; c_{s+t} \neq 0, \; t \geqq 1. \tag{13}$$

If $t = 1$, K is composed of the operators $[\delta_{ij}]$, $\delta_{ii-1} = 0$, $\delta_{s+1s-1} = 0$, the remaining δ_{ij} $(j < i-1)$ being arbitrary. If $t = 2$, then $\delta_{ii-1} = 0$, while the δ_{ij} $(j < i-1)$ are subject only to the condition.

$$c_s \delta_{s+1s-1} - c_{s+2} \delta_{s+2s} = 0. \tag{14}$$

If $t \geqq 3$, then $\delta_{ii-1} = 0$, while every δ_{ij} $(j < i-1)$ is arbitrary.

By §7, K contains every $B_{ij\alpha}$ $(j < i-1)$ except $B_{ii-2\alpha}$ $(i = s+1, \ldots, s+t)$. Next, K contains

$$(B_\alpha, B_\beta): \; \xi'_i = \xi_i + \sum_{j=i-2}^{s-1} \delta_{ij} \xi_j \qquad (i = s+1, \ldots, s+t) \tag{15}$$

where

$$\delta_{ii-2} = \beta_{ii-1} \alpha_{i-1i-2} - \alpha_{ii-1} \beta_{i-1i-2}. \tag{16}$$

By choice of the ρ_{ij} the product of (15) by $\prod_{i=s+2}^{s+t} \prod_{j=i-3}^{s-1} B_{ij\rho_{ij}}$ becomes

$$\xi'_i = \xi_i + \delta_{ii-2} \xi_{i-2} \qquad (i = s+1, \ldots, s+t). \tag{17}$$

Hence K contains every operator (17) subject to (16). Set

$$c'_j = c_{s+j}, \; \alpha_j = \alpha_{s+js+j-1}, \; \beta_j = \beta_{s+js+j-1}, \; \delta_j = \delta_{s+js+j-2} \; (j = 1, \ldots, t). \tag{18}$$

Then (16), (13), and $\Sigma c_i \beta_{ii-1} = 0$ become, respectively,*

$$\delta_j = \beta_j \alpha_{j-1} - \alpha_j \beta_{j-1} (j = 1, \ldots, t), \sum_{j=0}^{t} c'_j \alpha_j = 0, \sum_{j=0}^{t} c'_j \beta_j = 0. \tag{19}$$

Since $c'_0 \neq 0$, the last two equations serve to determine α_0 and β_0. Eliminating the latter from δ_1, we get

$$c'_0 \delta_1 = \sum_{i=2}^{t} (\alpha_1 \beta_i - \beta_1 \alpha_i) c'_i.$$

Hence $\delta_1 = 0$ if $t = 1$. For $t > 1$, $c'_0 \delta_1 - c'_2 \delta_2$ equals

$$\delta \equiv \sum_{i=3}^{t} (\alpha_1 \beta_i - \beta_1 \alpha_i) c'_i. \tag{20}$$

Hence $\delta = 0$ if $t = 2$, and we can make either δ_1 or δ_2 arbitrary, the other being determined by $c'_0 \delta_1 - c'_2 \delta_2 = 0$, viz. (14).

If $t > 2$, we proceed to show that $\delta, \delta_2, \ldots, \delta_t$ (and hence $\delta_1 \delta_2, \ldots, \delta_t$) can be made arbitrary by choice of the α_i, β_i $(i = 1, \ldots, t)$. Now $\delta_2, \ldots, \delta_t - 1$ involve neither β_t nor α_t, while in δ and δ_t the determinant of the coefficients of β_t and α_t is $-c'_t \Delta$, where

$$\Delta \equiv \alpha_1 \beta_{t-1} - \beta_1 \alpha_{t-1}.$$

Hence it suffices to make $\Delta \neq 0$ and $\delta_2, \ldots, \delta_{t-1}$ arbitrary.

For $t \parallel 2\tau$, $\tau > 1$, it suffices to take

$$\beta_1 = 0, \beta_{2j-1} = 1 (j = 2, 3, \ldots, \tau), \alpha_1 = 1, \alpha_{4l-1} = 1, \alpha_{4l+1} = 0 \, (l = 1, 2, \ldots).$$

Then

$$\Delta = 1, \ \delta_2 = \beta_2, \ \delta_{4j+2} = -\alpha_{4j+2}, \ \delta_{4j+1} = \alpha_{4j},$$
$$\delta_{4j} = \beta_{4j} - \alpha_{4j} \ (j > 0), \ \delta_{4j+3} = \alpha_{4j+2} - \beta_{4j+2}.$$

For $t = 2\tau + 1$, $\tau > 1$, it suffices to take

$$\beta_1 = 0, \ \alpha_1 = 1, \ \beta_{2j} = 1 (j = 2, \ldots, \tau), \alpha_{4j} = 0 \ (j > 0), \ \alpha_{4j+2} = 1.$$

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* The determinant of the coefficients of $\alpha_0, \alpha_1, \ldots, \alpha_t$ equals

$$\left(\prod_{j=1}^{t-1} \beta_j\right) \sum_{i=0}^{t} c'_i \beta_i = 0.$$

Then

$$\Delta = 1, \delta_2 = \beta_2, \delta_4 = \alpha_3, \delta_3 = \beta_3 - \alpha_3\beta_2, \delta_{4j} = \alpha_{4j-1},$$
$$\delta_{4j+1} = -\alpha_{4j+1}, \delta_{4j+2} = \alpha_{4j+1} - \beta_{4j+1}, \delta_{4j+3} = \beta_{4j+3} - \alpha_{4j+3} \ (j > 0).$$

Finally, for $t = 3, \Delta = \delta_2$. If $\delta_2 \neq 0$, it suffices to take $\alpha_1 = 1, \beta_1 = 0, \beta_2 = \delta_2$. If $\delta_2 = 0$, $\delta_3 \neq 0$, it suffices to take

$$\beta_1 = \beta_2 = 0, \ \alpha_2 = 1, \ \beta_3 = \delta_3, \ \alpha_1\delta_3 c_3' = \delta.$$

If $\delta_2 = \delta_3 = 0$, it suffices to take $\beta_1 = \beta_2 = \alpha_2 = 0$, $\beta_3 = 1$, $\alpha_1 c_3' = \delta$.

9. For $t = 1$, $\delta_{s+1s-1} = 0$ gives* $\beta_{s+1s}\alpha_{ss-1} = \alpha_{s+1s}\beta_{ss-1}$. Thus, if $p > 2$, (13) is additive with respect to $\alpha_{s+1s-1} - \frac{1}{2}\alpha_{s+1s}\alpha_{ss-1}$. Hence $[\alpha]^p = [x]$ has every $x_{ii-1} = 0$, $x_{s+1s-1} = 0$, and thus belongs to K. *For $p > 2$, $n = 1$, the $(p^{m-1} - 1)/(p-1)$ subgroups of order $p^{\mu-2}$ of $\{c_s\alpha_{ss-1} + c_{s+1}\alpha_{s+1s} = 0\}$, $c_s \neq 0$, $c_{s+1} \neq 0$, are*

$$\left\{c_s\alpha_{ss-1} + c_{s+1}\alpha_{s+1s} = 0, \ k_1(\alpha_{s+1s-1} - \tfrac{1}{2}\alpha_{s+1s}\alpha_{ss-1}) + \sum_{\substack{i=2 \\ i \neq s}}^{m} k_i\alpha_{ii-1} = 0\right\}. \quad (21)$$

10. For $t = 2$, $p > 2$, group (13) is additive with respect to

$$f_\alpha \equiv c_s\alpha_{s+1s-1} - c_{s+2}\alpha_{s+2s} - \tfrac{1}{2}c_s\alpha_{s+1s}\alpha_{ss-1} + \tfrac{1}{2}c_{s+2}\alpha_{s+2s+1}\alpha_{s+1s}. \quad (22)$$

Indeed, employing the values of $\gamma_{ii-1}, \gamma_{s+1s-1}, \gamma_{s+2s}$ from (3), we get

$$f_\gamma - f_\alpha - f_\beta = \tfrac{1}{2}(c_s\delta_{s+1s-1} - c_{s+2}\delta_{s+2s}) = 0.$$

Further, if $[\alpha]^p$ be written $[\delta]$, then each $\delta_{ii-1} = 0$ and (14) holds, so that it belongs to K. *For $t = 2$, $p > 2$, $n = 1$, the $(p^{m-1} - 1)/(p-1)$ subgroups of order $p^{\mu-2}$ of (13) are*

$$\left\{\sum_{i=s}^{s+2} c_i\alpha_{ii-1} = 0, k_1 f_\alpha + \sum_{\substack{i=2 \\ i \neq s}}^{m} k_i\alpha_{ii-1} = 0\right\}. \quad (23)$$

11. *For $t > 2$, $n = 1$, the $(p^{m-2} - 1)/(p-1)$ subgroups of order $p^{\mu-2}$ of (13) are*

$$\left\{\sum_{i=s}^{s+t} c_i\alpha_{ii-1} = 0, \ \sum_{\substack{i=2 \\ i \neq s}}^{m} k_i\alpha_{ii-1} = 0\right\}. \quad (24)$$

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* Or directly from $c_s\alpha_{ss-1} + c_{s+1}\alpha_{s+1s} = 0$, $c_s\beta_{ss-1} + c_{s+1}\beta_{s+1s} = 0$, $c_s \neq 0$, $c_{s+1} \neq 0$.

12. For $n=1, p>2$, all the subgroups of order $p^{\mu-2}$ of G_{p^μ} have been determined in §§6, 9, 10, 11. The distinct ones are (21) and (23) for $k_1=1$, and

$$\left\{\sum_{i=2}^{m} c_i\alpha_{ii-1}=0,\ \sum_{i=2}^{m} k_i\alpha_{ii-1}=0\right\},\quad D_{ij}\equiv\begin{vmatrix} c_i & c_j \\ k_i & k_j\end{vmatrix}\ \text{not all zero,} \tag{25}$$

$$\left\{\alpha_{ss-1}=0,\ \sum_{\substack{i=2,\ldots,m\\ i\neq s}} k_i\alpha_{ii-1}+k_s\alpha_{ss-2}+k_1\alpha_{s+1s-1}=0\right\},\ k_s, k_1 \text{ not both } 0, \tag{26}$$

where $k_s\alpha_{ss-2}$ is to be suppressed if $s=2$, and $k_1\alpha_{s+1s-1}$ if $s=m$.

13. If $k_{s-1}=0$ in (26), we transform by

$$T_{s-1s}:\quad \xi'_{s-1}=\xi_s,\ \xi'_s=-\xi_{s-1}, \tag{27}$$

and get a group of the form (26) with $k_s=0$. If $k_{s-1}\neq 0$, the same result follows by transforming (26) by $B_{s-1s\rho}$, $\rho=k_s/k_{s-1}$. Consider (26) with $k_s=0$, $k_1=-1$, as we may set. If also $k_{s-1}=0$, $k_{s+1}\neq 0$, we transform by $B_{s-1s\rho}$, $\rho=k_{s+1}^{-1}$, and obtain a group (25). If $k_{s-1}=k_{s+1}=0$, we transform by (27) and obtain a group (25). If $k_{s-1}\neq 0$, we may make $k_{s-1}=1$ by transforming by

$$\xi'_{s-2}=k_{s-1}^{-1}\xi_{s-2},\ \xi'_s=k_{s-1}\xi_s.$$

Then transforming by $B_{ss-1\rho}$, $\rho=k_{s+1}$, we obtain

$$\left\{\alpha_{ss-1}=0,\ \alpha_{s+1s-1}=\sum_{\substack{i=2,\ldots,m\\ i\neq s,\, s+1}} \varkappa_i\alpha_{ii-1}\right\},\ \varkappa_{s-1}=1. \tag{28}$$

If $s=2$, α_{s-1s-2} is to be suppressed. If $s=m$, α_{s+1s-1} is to be replaced by zero and the group falls under (25).

In (21) or (23) with $k_1=1$, we may make $k_{s+1}=0$ by transforming by $B_{ss-1\rho}$, so that f_a is replaced by $f_a-\rho c_s\alpha_{s+1s}$.

14. Consider the commutator subgroup K of the group H defined by (28) for $s<m$. H contains $B_{s+1s\alpha}$, $B_{ii-2\alpha}$ $(i\neq s+1)$, $B_{ij\alpha}$ $(j<i-2)$,

$$C_\alpha\equiv B_{s+1s-1\alpha}B_{s-1s-2\alpha},\ A_{i\alpha}\equiv B_{ii-1\alpha}B_{s+1s-1\kappa_i\alpha},\ B_{i\alpha}\equiv B_{s-1s-2-\kappa_i\alpha}B_{ii-1\alpha},$$

for $i\neq s-1, s, s+1$. Hence K contains

$(B_{ii-3\alpha}, B_{i-3j\beta})=B_{ij\gamma}$ $(j\overline{\gtrless} i-6)$, $(B_{ii-2\alpha}, B_{i-2i-4\beta})=B_{ii-4\gamma}$ $(i\neq s+1, s+3)$, $(B_{ii-3\alpha}, B_{i-3i-5\beta})=B_{ii-5\gamma}$ $(i\neq s+4)$, $(B_{ii-2\alpha}, B_{i-2i-5\beta})=B_{ii-5\gamma}$ $(i\neq s+1)$,

$\gamma = -\alpha\beta$. By the last two, K contains every $B_{ii-5\gamma}$. Also

$$(B_{s-1s-3\rho}, C_\alpha) = B_{s+1s-3\alpha\rho}, \; (B_{s+3s+1\rho}, C_\alpha) = B_{s+3\;-1-\alpha\rho} B_{s+3s-2\alpha^2\rho},$$
$$(A_{i\alpha}, B_{i-1i-3\beta}) = B_{ii-3\gamma} (i \neq s-1, s, s+1, s+2, s+4),$$
$$(A_{s-2\alpha}, B_{ss-2\beta}) = B_{ss-3\alpha\beta},$$
$$(B_{s+1s\alpha}, B_{ss-2\beta}) = B_{s+1s-2\gamma}, \; (B_{s-2s-4\alpha}, C_1) = B_{s-1s-4\alpha},$$
$$(A_{s+4\alpha}, B_{s+3s+1\beta}): \quad \xi'_{s+4} = \xi_{s+4} - \alpha\beta\xi_{s+1} + \varkappa\alpha^2\beta\xi_{s-1},$$
$$\xi'_{s+3} = \xi_{s+3} + \varkappa\alpha\beta\xi_{s-1},$$
$$(A_{s+2\alpha}, C_1): \quad \xi'_{s+2} = \xi_{s+2} - \alpha\xi_{s-1} + \alpha\xi_{s-2}, \; \xi'_{s+1} = \xi_{s+1} - \varkappa\alpha\xi_{s-2}.$$

Hence K contains every $B_{ii-4\gamma}$, $B_{ii-3\gamma}$. Also

$$(B_{i\alpha}, B_{i-11}) = B_{ii-2-\alpha} \; (i \neq s-2, s-1, s, s+1),$$
$$(A_{s-2\alpha}, A_{s-31}) = B_{s-2s-4-\alpha}$$
$$(C_\alpha, A_{s-21}) = \xi'_{s+1} = \xi_{s+1} + \varkappa\alpha\xi_{s-2} - \varkappa\alpha\xi_{s-3}, \; \xi'_{s-1} = \xi_{s-1} - \alpha\xi_{s-3}.$$

Hence K contains every $B_{ii-2\gamma}$ $(i \neq s, s+1)$. But in every transformation $[\delta_{ij}]$ of K, $\delta_{ii-1} = 0$, $\delta_{ss-2} = 0$, $\delta_{s+1s-1} = 0$. We have shown that the remaining δ_{ij} $(j < i)$ may be chosen arbitrarily. Further, $[\alpha]^p$ belongs to K. Also, H is additive with respect to α_{ii-1}, α_{ss-2}, α_{s+1s-1}.

THEOREM. *For $n = 1$, $2 < s < m$, the $(p^{m-1} - 1)/(p-1)$ subgroups of index p of (28) are given by annexing a linear relation between α_{ss-2}, $\alpha_{ii-1} (i = 2, \ldots, m; i \neq s)$. For $s = 2$, the $(p^{m-2} - 1)/(p-1)$ subgroups are given by annexing a linear relation between the $\alpha_{ii-1} = 0$ $(i \neq s)$.*

15. We may show similarly that the commutator subgroup of (23) with $k_1 = 1$, $k_{s+1} = 0$ (see end of §13) is composed of the $[\delta_{ij}]$ with $\delta_{ii-1} = 0$, and the δ_{ij} $(j < i-1)$ subject only to (14), and that it contains every $[\alpha]^p$. Hence, if $n = 1$, there are $(p^{m-2} - 1)/(p-1)$ subgroups of index p given by annexing a linear relation between the α_{ii-1}.

The commutator subgroup of (21) with $k_1 = 1$, $k_{s+1} = 0$ has $\delta_{ii-1} = 0$, $\delta_{s+1s-1} = 0$ and the remaining δ_{ij} $(j < i-1)$ arbitrary. Hence, if $n = 1$, there are $(p^{m-2} - 1)/(p-1)$ subgroups of index p given by annexing a linear relation between the α_{ii-1}.

I have not completed the longer discussion necessary for (25).

16. In the general problem special treatment is necessary for

$$\{\alpha_{s_1 s_1-1}=0,\ \alpha_{s_2 s_2-1}=0,\ \ldots,\ \alpha_{s_r s_r-1}=0\},\ s_1<s_2<\ldots<s_r. \tag{29}$$

The commutator group K of (29) is formed of the operators $[\delta_{ij}]$ with $\delta_{ii-1}=0$, $\delta_{jj-2}=0$, where j ranges over the distinct numbers of the set

$$s_1,\ s_2,\ \ldots,\ s_r,\ s_1+1,\ s_2+1,\ \ldots,\ s_r+1, \tag{30}$$

$\delta_{ii-3}=0$ $(i=\sigma_1, \ldots, \sigma_\rho)$, where $\sigma_1, \ldots, \sigma_\rho$ denote the distinct numbers such that both σ_i and σ_i-2 belong to the set $s_1, \ldots, s_r$, while the remaining δ_{ij} are arbitrary. In proof, (29) contains every $B_{ij\alpha}(j<i)$ except $B_{s_i s_i-1\alpha}$ $(i=1, \ldots, r)$. Then by (8), K contains every $B_{ij\gamma}$ $(j<i-3)$ and every $B_{ii-2\gamma}$, i not in the set (30). By (9), K contains every $B_{ii-3\gamma}$, $i\neq\sigma_1, \ldots, \sigma_\rho$. Moreover, (29) is additive with respect to the α_{jj-2}, j in (30), and the α_{ii-3}, $i=\sigma_1, \ldots, \sigma_\rho$. Hence K has the form given. It follows that, *if* $n=1$, *the subgroups of index* p *of* (29) *are obtained by annexing a linear relation between the* α_{ii-1} $(i\neq s_1, \ldots, s_r)$, *the* α_{jj-2}, j *in the set* (30), *and* α_{ii-3} $(i=\sigma_1, \ldots, \sigma_\rho)$.

Subgroups of order a power of p of SLH $(3, p^n)$.

17. We consider the subgroups of $G_{p^{3n}}$. The commutator

$$([\alpha],\ [\beta])=B_{3,1,\delta},\ \delta=\beta_{32}\alpha_{21}-\alpha_{32}\beta_{21}.$$

By §5, each of the $(p^{2n}-1)/(p-1)$ subgroups of order p^{3n-1} is defined by

$$f_1(a)=\sum_{k=0}^{n-1}(\lambda_k a_{21k}+\mu_k a_{32k})\equiv 0 \pmod p.$$

If $n>1$, the commutator subgroup of any of these $G_{p^{3n-1}}$ is formed of the p^n operators $B_{3,1,\delta}$. In proof, we show that δ may be made arbitrary. If $\lambda_i\neq 0$, we take $\alpha_{32}=0$, $a_{21j}=1$, j being a particular subscript $\neq i$. Then $\alpha_{21}\neq 0$ in view of the irreducibility of (4). Hence we can choose β_{32} to make δ arbitrary and then determine β_{21} to make $f_1(b)\equiv 0$. If every $\lambda_k=0$, we take $\beta_{32}=0$, the a_{32k} such that $\alpha_{32}\neq 0$ and $f_1(a)=0$, and determine β_{21} to make δ arbitrary. *Hence, if* $n>1$, *the* $(p^{2n-1}-1)/(p=1)$ *subgroups of index* p *of* $G_{p^{3n-1}}$ *are given by*

$$f_1(a)\equiv 0,\quad f_2(a)\equiv 0.$$

Consider such a subgroup $G_{p^{3n-2}}$ for $n > 2$. If the determinant of the coefficients of a_{21i} and a_{21j} in $f_1(a)$ and $f_2(a)$ is not $\equiv 0 \pmod p$, we take $\alpha_{32} = 0$, $a_{21l} = 1$, l being a particular subscript different from i and j; then $\alpha_{21} \not\equiv 0$ and we may determine β_{32} to make δ arbitrary in the $GF[p^n]$, and b_{21i} and b_{21j} to make $f_1(b) \equiv 0$, to $f_2(b) \equiv 0$. A similar result follows if the determinant of the coefficients of a_{32i} and a_{32j} is not $\equiv 0 \pmod p$. There remains the case

$$\left\{\sum_{k=0}^{n-1} \lambda_k a_{21k} \equiv 0, \quad \sum_{k=0}^{n-1} l_k a_{32k} \equiv 0\right\}.$$

Now $\xi_1' = s^{-1}\xi_1$, $\xi_3' = t\xi_3$ transforms $[\alpha]$ into $[\alpha']$ where

$$\alpha_{21}' = s\alpha_{21}, \quad \alpha_{32}' = t\alpha_{32}, \quad \alpha_{31}' = st\alpha_{31}.$$

Let $s = \Sigma s_k\rho^k$, $\alpha_{21}' = \Sigma a_{21k}'\rho^k$. Then a_{21k}' is a bi-linear form in s_k, a_{21k} $(k = 0, 1, \ldots, n-1)$ of determinant* not $\equiv 0 \pmod p$. Hence we may choose s so that a_{21n-1}' shall be identical with $\Sigma\lambda_k a_{21k}$. Proceeding similarly with α_{32}', the group becomes $\{a_{21n-1} = a_{32n-1} = 0\}$. For the latter we take $a_{210} = 1$, $a_{21k} = 0\ (k > 0)$, $b_{210} = 0$, $b_{211} = 1$, $b_{21k} = 0\ (k > 1)$, $b_{32k} = 0\ (k > 0)$. Then

$$\delta = b_{320} - \sum_{k=0}^{n-2} a_{32k}\rho^{k+1}$$

is arbitrary in the $GF[p^n]$. *Hence the commutator subgroup of $G_{p^{3n-2}}$ for $n > 2$ is of order p^n; the $(p^{2n-2} - 1)/(p - 1)$ subgroups of index p of $G_{p^{3n-2}}$ are defined by three linear congruences $f_1(a) \equiv 0$, $f_2(a) \equiv 0$, $f_3(a) \equiv 0$.*

Consider such a subgroup $G_{p^{3n-3}}$ for $n > 3$. If the determinant of the coefficients of a_{21i}, a_{21j}, a_{21l} is not $\equiv 0 \pmod p$, we take $\alpha_{32} = 0$, $a_{21r} \equiv 1$, r being a particular subscript different from i, j, l; then $\alpha_{21} \not\equiv 0$, and we can determine β_{32} to make δ arbitrary in the $GF[p^n]$ and b_{21i}, b_{21j}, b_{21l} to make $f_1(b) \equiv 0$, $f_2(b) \equiv 0$, $f_3(b) \equiv 0$. If all such determinants are $\equiv 0 \pmod p$ and likewise for the determinants of the coefficients of a_{32i}, the three relations become

$$\sum_{k=0}^{n-1} m_k a_{21k} \equiv 0, \ \sum_{k=0}^{n-1} l_k a_{32k} \equiv 0, \ f_3(a) \equiv 0.$$

As before we normalize by transformation and reach

$$\left\{a_{21n-1} \equiv 0, \ a_{32n-1} \equiv 0, \ f_3(a) = \sum_{k=0}^{n-2} (\lambda_k a_{21k} + \mu_k a_{32k}) \equiv 0\right\}$$

* *A general theorem on algebraic numbers*, Bull. Amer. Math. Soc., vol. 11 (1905), pp. 482-486.

We proceed to show that in $\delta = \Sigma d_k \rho^k$ the d_k may be made arbitrary mod p. In view of the symmetry, we may assume that the μ_k are not all zero. We first take $\alpha_{21} = 1$, $\beta_{21} = \rho$. Then

$$d_0 = b_{320},\ d_{n-1} = -a_{32n-2},\ d_k = b_{32k} - a_{32k-1}\ (k = 1, \ldots, n-2).$$

The conditions $f_3(a) \equiv 0$, $f_3(b) \equiv 0$ become, respectively,

$$\begin{array}{cccc|l} a_{320} & a_{321} & a_{322} \cdots a_{32n-3} & & \\ \hline \mu_0 & \mu_1 & \mu_2 \cdots \mu_{n-3} & & = -\lambda_0 + \mu_{n-2} d_{n-1} \\ \mu_1 & \mu_2 & \mu_3 \cdots \mu_{n-2} & & = -\lambda_1 - \mu_0 d_0 - \sum_{k=1}^{n-2} \mu_k d_k . \end{array}$$

They may be satisfied by choice of the a_{32k} unless every

$$\begin{vmatrix} \mu_i & \mu_j \\ \mu_{i+1} & \mu_{j+1} \end{vmatrix} \equiv 0 \qquad (i, j = 0, 1, \ldots, n-3). \tag{31}$$

We next take $\alpha_{21} = 1$, $\beta_{21} = \rho^2$. Then, applying (4),

$d_0 = b_{320} - a_{32n-2} r_0$, $d_1 = b_{321} - a_{32n-2} r_1$,
$d_{n-1} = -a_{32n-3} - a_{32n-2} r_{n-1}$, $d_k = b_{32k} - a_{32k-2} - a_{32n-2} r_k$ $(k = 2, \ldots, n-2)$.
Conditions $f_3(a) \equiv 0$, $f_3(b) \equiv 0$ become, respectively,

$$\begin{array}{ccc|l} a_{320} & a_{321} \cdots a_{32n-4} & a_{32n-2} & \\ \hline \mu_0 & \mu_1 \cdots \mu_{n-4} & \mu_{n-2} - \mu_{n-3} r_{n-1} & = -\lambda_0 + \mu_{n-3} d_{n-1} \\ \mu_2 & \mu_3 \cdots \mu_{n-2} & \sum_{k=0}^{n-2} \mu_k r_k & = -\lambda_2 - \sum_{k=0}^{n-2} \mu_k d_k . \end{array}$$

They may be satisfied by choice of the a_{32k} unless every

$$\begin{vmatrix} \mu_i & \mu_j \\ \mu_{i+2} & \mu_{j+2} \end{vmatrix} \equiv 0, \quad D_i = \begin{vmatrix} \mu_i & \mu_{n-2} - \mu_{n-3} r_{n-1} \\ \mu_{i+2} & \sum_{k=0}^{n-2} \mu_k r_k \end{vmatrix} \equiv 0, \tag{32}$$

for $i, j = 0, 1, \ldots, n-4$. Let, therefore, (31) and (32) all hold. If $\mu_0 \equiv 0$, then $\mu_1 \equiv 0, \ldots, \mu_{n-3} \equiv 0$ by (31) and $\mu_{n-2} \equiv 0$ by $D_{n-4} \equiv 0$, whereas not every $\mu_k \equiv 0$ by hypothesis. Hence $\mu_0 \not\equiv 0$. Then in $f_3(a) \equiv 0$, we may set $\mu_0 \equiv 1$. Then by (31), $\mu_k \equiv \mu_1^k$ $(k = 1, \ldots, n-2)$. Then

$$D_0 \equiv \sum_{k=0}^{n-1} \mu_1^k r_k - \mu_1^n \equiv 0,$$

so that (4) has the root $\rho = \mu_1$, contrary to its irreducibility. Hence, if $n > 3$, the commutator subgroup of $G_{p^{3n-3}}$ is of order p^n. *For $n > 3$, the $(p^{2n-3}-1)/(p-1)$ subgroups of index p of $G_{p^{3n-3}}$ are obtained by annexing a fourth linear relation $f_4(a) \equiv 0$.*

I do not enter upon further details here of the proof of the theorem: *For $r \overline{\gtrless} n$, any subgroup of order p^{3n-r} of $G_{p^{3n}}$ is defined by r independent linear homogeneous congruences between the a_{21k}, a_{32k} $(k = 0, 1, \ldots, n-1)$.*

18. I will here express my belief in the truth of the following general theorem: *For $r \overline{\gtrless} n$, any subgroup of order $p^{\mu n - r}$ of $G_{p^{\mu n}}$ is defined by r independent linear homogeneous congruences between the a_{ii-1k} $(i = 2, \ldots, m;$ $k = 0, 1, \ldots, n-1)$.* Such a theorem would include the results of §§5, 6, 17, 20.

19. We proceed to determine the subgroups of order p^3 of $SLH(3, p^2)$. For (4) we take $\rho^2 \equiv \mu\rho + \nu \pmod p$. Set $\delta = d + D\rho$. Then

$$d \equiv \Delta_{00} + \nu\Delta_{11},\ D \equiv \Delta_{01} + \Delta_{10} + \mu\Delta_{11},\ \Delta_{ij} = b_{32i}a_{21j} - a_{32i}b_{21j}.$$

Any subgroup G_{p^4} of G_{p^6} is defined (§17) by two independent congruences $f_1(a) \equiv 0$, $f_2(a) \equiv 0$. If the determinant of the coefficients of a_{210} and a_{211} is $\not\equiv 0$, we get

$$\{a_{210} \equiv g a_{320} + h a_{321},\ a_{211} \equiv l a_{320} + m a_{321}\}. \tag{33}$$

In the contrary case, we obtain one of the following:

$$\{a_{321} = h a_{320},\ a_{211} = l a_{210} + m a_{320}\},\ \{a_{321} = h a_{320},\ a_{210} = m a_{320}\}, \tag{34}$$

$$\{a_{320} = 0,\ a_{211} = l a_{210} + m a_{321}\},\ \{a_{320} = 0,\ a_{210} = m a_{321}\},\ \{\alpha_{32} = 0\}. \tag{35}$$

For the five groups (33)-(35), d and D are respectively

$$(h - l\nu)(b_{320}a_{321} - a_{320}b_{321}),\ (m - g - l\mu)(b_{320}a_{321} - a_{320}b_{321}); \tag{33$'$}$$

$$(1 + hl\nu)\Delta_{00},\ (h + l + hl\mu)\Delta_{00};\ h\nu\Delta_{01},\ (1 + h\mu)\Delta_{01}; \tag{34$'$}$$

$$l\nu\Delta_{10},\ (1 + l\mu)\Delta_{10};\ \nu\Delta_{11},\ \mu\Delta_{11};\ 0,\ 0. \tag{35$'$}$$

The a's and b's entering (33)$'$-(35)$'$ are arbitrary. Now d and D are both identically zero only for $\{\alpha_{32} = 0\}$ and for (33) with the two coefficients in (33)$'$ vanishing so that (33) becomes

$$\{\alpha_{21} = \tau\alpha_{32}\},\quad \tau = g + \rho h \nu^{-1}. \tag{36}$$

The statement is evident except for $(34)_1$. But if $1+hl\nu=0$ and $h+l+hl\mu=0$, then $-h^{-1}$ and $-l^{-1}$ are the roots of $\rho^2-\mu\rho-\nu\equiv 0$, contrary to the irreducibility of (4). *The only commutative* G_{p^4} *are* $\{\alpha_{32}=0\}$ and $\{\alpha_{21}=\tau\alpha_{32}\}$; *the commutator subgroups of the remaining* G_{p^4} *are of order* p.

If $p>2$, every ternary $[\alpha]$ is of period p. The number of subgroups G_{p^3} of each G_{p^4} now follows from §4. They may be obtained by employing $(33')$-$(35)'$.

If $p>2$, *the subgroups* G_{p^3} *of the* G_{p^4} *given in the first column are obtained by annexing to the two relations defining the* G_{p^4} *an independent linear homogeneous relation between the elements given in the same line of the second column*:

$(33)=(36)$	$a_{320},\ a_{321},\ a_{310}-\frac{1}{2}ga_{320}^2-ha_{320}a_{321}-\frac{1}{2}(\nu g+\mu h)\,a_{321}^2,$ $a_{311}-\frac{h}{2\nu}a_{320}^2-(g+\frac{\mu h}{\nu})\,a_{320}a_{321}-\frac{1}{2}(h+\frac{h\mu^2}{\nu}+\mu g)\,a_{321}^2$
$(33)\neq(36)$	$a_{320},\ a_{321},\ (m-g-l\mu)\,a_{310}-(h-\nu l)\,a_{311}-ua_{320}^2-va_{320}a_{321}-wa_{321}^2$
$(34)_1$	$a_{320},\ a_{210},\ (l+h+lh\mu)\,a_{310}-(1+hl\nu)\,a_{311}+\frac{m}{2}(1+h\mu-h^2\nu)\,a_{320}^2$
$(34)_2$	$a_{320},\ a_{211},\ (1+h\mu)\,a_{310}-h\nu a_{311}-\frac{m}{2}(1+h\mu-h^2\nu)\,a_{320}^2$
$(35)_1$	$a_{321},\ a_{210},\ (1+l\mu)\,a_{310}-l\nu a_{311}-\frac{1}{2}m\nu a_{321}^2$
$(35)_2$	$a_{321},\ a_{211},\ \mu a_{310}-\nu a_{311}+\frac{1}{2}m\nu a_{321}^2$
$\{\alpha_{32}=0\}$	$a_{210},\ a_{211},\ a_{310},\ a_{311},$

where for the second group occur the abbreviations

$$u=\tfrac{1}{2}g\,(m-g-l\mu)-\tfrac{1}{2}l\,(h-\nu l),\ v=m\nu l-hg-hl\mu,$$
$$w=\tfrac{1}{2}m\nu\,(m-g-l\mu)-\tfrac{1}{2}\,(h-\nu l)(h+m\mu).$$

Subgroups of order a power of p in SLH$(4,p^2)$.

20. Let the $GF[p^2]$ be defined by the irreducible congruence $\rho^2\equiv\mu\rho+\nu \pmod p$. The commutator of $[\alpha]$ and $[\beta]$ is $[\delta]$, where $\delta_{i\,i-1}=0$,

$$\delta_{31}=\beta_{32}\alpha_{21}-\alpha_{32}\beta_{21},\ \delta_{42}=\beta_{43}\alpha_{32}-\alpha_{43}\beta_{32},$$
$$\delta_{41}=\beta_{43}\alpha_{31}-\alpha_{43}\beta_{31}+\beta_{42}\alpha_{21}-\alpha_{42}\beta_{21}-\delta_{42}\,(\alpha_{21}+\beta_{21}).$$

Set $\alpha_{ij}=a_{ij}+\rho A_{ij}$, $\delta_{ij}=d_{ij}+\rho D_{ij}$. By §5, any subgroup of order p^{11} of $G_{p^{12}}$ is defined by a relation

$$\sum_{i=2}^{4}c_i a_{i\,i-1}+\sum_{i=2}^{4}k_i A_{i\,i-1}\equiv 0 \pmod p. \tag{37}$$

The commutator subgroup K of $G_{p^{11}}$ contains $B_{41\delta}$, δ arbitrary (§7). If c_3 and k_3 are not both zero, we can take $\alpha_{32} \neq 0$, $\alpha_{21} = \alpha_{43} = 0$, β_{21} and β_{43} arbitrary, so that δ_{31} and δ_{42} can be made arbitrary. To accomplish the latter when $c_3 = k_3 = 0$, we can take $\alpha_{43} = 0$, $\alpha_{21} \neq 0$, $\beta_{43} \neq 0$, α_{32} and β_{32} arbitrary; the determinant of the coefficients of the latter in δ_{31} and δ_{42} is then $\alpha_{21}\beta_{43} \neq 0$. Hence K is of order p^6. The $(p^5 - 1)/(p - 1)$ subgroups of order p^{10} of $G_{p^{11}}$ are obtained by annexing an independent linear relation

$$\sum_{i=2}^{4} d_i a_{ii-1} + \sum_{i=2}^{4} l_i A_{ii-1} \equiv 0 \pmod{p}. \tag{38}$$

21. Consider the commutator subgroup K of $G_{p^{10}}$ defined by (37) and (38). We assume that these are not equivalent to $a_{ss-1} = 0$, the contrary case having been treated in §6. Then, by §7, K contains $B_{41\delta}$, δ arbitrary in the $GF[p^2]$. For brevity we write $(b_{32}a_{21})$ for $b_{32}a_{21} - a_{32}b_{21}$, etc. Then

$$d_{31} = (b_{32}a_{21}) + \nu(B_{32}A_{21}),\ D_{31} = (b_{32}A_{21}) + (B_{32}a_{21}) + \mu(B_{32}A_{21}),$$
$$d_{42} = (b_{43}a_{32}) + \nu(B_{43}A_{32}),\ D_{42} = (b_{43}A_{32}) + (B_{43}a_{32}) + \mu(B_{43}A_{32}).$$

Let first $c_2 = k_2 = d_2 = l_2 = 0$. If $c_3l_3 - d_3k_3 \neq 0$, the group is

$$\{a_{32} = ra_{43} + sA_{43},\ A_{32} = ta_{43} + uA_{43}\}, \tag{39}$$

r, s, t, u, not all zero. If $c_3l_3 - d_3k_3 = 0$, we obtain one of the following:

$$\{A_{43} = ra_{43},\ A_{32} = sa_{32} + ta_{43}\},\ \{A_{43} = ra_{43},\ a_{32} = ta_{43}\}, \tag{40}$$
$$\{a_{43} = 0,\quad A_{32} = sa_{32} + tA_{43}\},\ \{a_{43} = 0,\quad a_{32} = tA_{43}\}. \tag{41}$$

For (39) we have

$$d_{31} = r(b_{43}a_{21}) + s(B_{43}a_{21}) + \nu t(b_{43}A_{21}) + \nu u(B_{43}A_{21}),$$
$$D_{31} = t(b_{43}a_{21}) + u(B_{43}a_{21}) + (r + \mu t)(b_{43}A_{21}) + (s + \mu u)(B_{43}A_{21}),$$
$$d_{42} = (s - \nu t)(b_{43}A_{43}),\ D_{42} = (u - r - \mu t)(b_{43}A_{43}).$$

We may give to d_{31}, D_{31}, d_{42}, D_{42}, any values such that

$$(u - r - \mu t)\, d_{42} = (s - \nu t)\, D_{42}. \tag{39}$$

Indeed, if both of the determinants

$$\begin{vmatrix} r & \nu t \\ t & r + \mu t \end{vmatrix}, \qquad \begin{vmatrix} s & \nu u \\ u & s + \mu u \end{vmatrix},$$

vanish, then from the irreducibility of $\rho^2-\mu\rho-\nu$ would r, t, s, u, all vanish. If the coefficients in the $(39)'$ are zero, (39) becomes

$$\{a_{32}=\tau a_{43}\}, \qquad \tau=u-\mu t+\rho t. \tag{42}$$

According as (39) *is or is not of the form* (42), *its commutator subgroup is of order* p^4 *or* p^5. For each of the groups (40), (41), the discussion is similar but simpler, so that only the results need be given. *The commutator subgroup of each of the groups* (40), (41) *is of order* p^5, the d_{ij} and D_{ij} having any values subject to the respective conditions

$$(s+r+\mu rs)\,d_{42}=(1+\nu rs)\,D_{42}, \quad (1+\mu r)\,d_{42}=\nu rD_{42}, \tag{40$'$}$$
$$(1+\mu s)\,d_{42}=\nu sD_{42}, \qquad \mu d_{42}=\nu D_{42}. \tag{41$'$}$$

Let next $c_2l_2-d_2k_2\neq 0$. Then (37) and (38) may be written

$$a_{21}=ra_{43}+sA_{43}+ja_{32}+kA_{32}, \quad A_{21}=ta_{43}+wA_{43}+la_{32}+mA_{32}.$$

We obtain the following values of the d_{ij}, D_{ij}:

	$(b_{32}a_{43})$	$(b_{32}A_{43})$	$(b_{32}A_{32})$	$(B_{32}a_{43})$	$(B_{32}A_{43})$
$d_{31}=$	r	s	$k-\nu l$	νt	νw
$D_{31}=$	t	w	$m-j-\mu l$	$r+\mu t$	$s+\mu w$
$d_{42}=$	-1	0	0	0	$-\nu$
$D_{42}=$	0	-1	0	-1	$-\mu$

It suffices to discuss these equations with $(b_{32}a_{43})$, ..., as variables; for, taking $(b_{32}A_{32})=1$, we may determine a_{43} and b_{43} from $(b_{32}a_{43})$ and $(B_{32}a_{43})$, A_{43} and B_{43} from $(b_{32}A_{43})$ and $(B_{32}A_{43})$. Hence K will be of order p^6 unless every determinant of the fourth order in the above matrix is zero. That of the 1st, 2nd, 4th, 5th columns equals

$$-(\nu t-s)^2-\mu(\nu t-s)(r-w+\mu t)+\nu(r-w+\mu t).^2$$

In view of the irreducibility of (4), this vanishes only if

$$s=\nu t,\ r=w-\mu t. \tag{43}$$

When (43) holds every determinant of order 4 vanishes, and

$$\begin{aligned} d_{31}(m-j-\mu l)+D_{31}(\nu l-k)+d_{42}\,[(w-\mu t)(m-j-\mu l)-t\,(k-\nu l)] \\ +D_{42}\,[\nu t\,(m-j-\mu l)-w\,(k-\nu l)]=0. \end{aligned} \tag{44}$$

If also
$$m-j-\mu l=0,\ k-\nu l=0, \tag{45}$$

then all determinants of order 3 in the matrix vanish identically, and we obtain

$$D_{31} + td_{42} + wD_{42} = 0,\ d_{31} + (w - \mu t)\, d_{42} + vtD_{42} = 0. \tag{46}$$

When conditions (43) *and* (45) *all hold, the group becomes*

$$\{\alpha_{21} = (r + \rho t)\,\alpha_{43} + (j + \rho l)\,\alpha_{32}\}, \tag{47}$$

and its commutator subgroup K is of order p^4. When (43) *but not* (45) *hold,* K is of order p^5. When* (43) *do not hold, K is of order p^6.*

Let finally $c_2 l_2 - d_2 k_2 = 0$, but c_2, l_2, d_2, k_2, not all zero. Then (37) and (38) may be written $ca_{21} + kA_{21} + f_1 \equiv 0, f_2 \equiv 0$, where f_1 and f_2 are linear functions of $a_{32}, a_{43}, A_{32}, A_{43}$, and c, k are not both zero. *If the coefficient of either A_{43} or a_{43} in f_2 is not zero, I find that the commutator subgroup K is of order p^6*, each $\delta_{ij}\ (j < i - 1)$ being arbitrary. There remain the groups

$$\{A_{32} = la_{32},\ a_{21} = \phi\},\ \{A_{32} = la_{32},\ A_{21} = ca_{21} + \phi\},\ \phi \equiv rA_{43} + sa_{43} + ka_{32}, \tag{48}$$

$$\{a_{32} = 0,\quad a_{21} = \psi\},\ \{a_{32} = 0,\quad A_{21} = ca_{21} + \psi\},\ \psi \equiv rA_{43} + sa_{43} + kA_{32}. \tag{49}$$

For each of these groups (48), (49), *K is of order p^5*, the d_{ij}, D_{ij} being subject to a single condition:

$$(1 + \mu l)\, d_{31} - vlD_{31} + (r - vls)\, D_{42} + (s + \mu ls - lr)\, d_{42} \equiv 0, \tag*{$(48)_1'$}$$

$$(1 + vlc)\, D_{31} - (c + l + \mu lc)\, d_{31} + (r - vls)\, D_{42} + (s + \mu ls - lr)\, d_{42} \equiv 0, \tag*{$(48)_2'$}$$

$$vD_{31} - \mu d_{31} + vsD_{42} + (r - \mu s)\, d_{42} \equiv 0, \tag*{$(49)_1'$}$$

$$(1 + \mu c)\, d_{31} - vcD_{31} + vsD_{42} + (r - \mu s)\, d_{42} \equiv 0. \tag*{$(49)_2'$}$$

The largest m-ary linear group containing self-conjugately a given subgroup of order a power of p.

22. *The $p^{\mu n}(p^n - 1)^m$ operators transforming $G_{p^{\mu n}}$ into itself are*

$$(\delta_{ij}),\ \delta_{ij} = 0\ (j > i),\ \delta_{ii} \neq 0. \tag{50}$$

Let (δ_{ij}) be a general matrix. Equating the coefficients of ξ_j in the functions by which $[A_{ij}](\delta_{ij})$ and $(\delta_{ij})[\alpha_{ij}]$ replace ξ_i, we get

$$\sum_{k=j+1}^{m} \delta_{ik} A_{kj} = \sum_{l=1}^{i-1} \alpha_{il}\delta_{lj} (i, j = 1, \ldots, m). \tag{51}$$

* Then $a_{21} = (r + \rho t)\, a_{43} + xa_{32} + yA_{32}$, $y \neq \rho x$.

The $A_{ij}\,(j<i)$ may be taken arbitrary, while the α_{ij} are then to be determined. For $i=j=1$, (51) gives $\delta_{1k}=0\,(k=2,\ldots,m)$. To proceed by induction, suppose that $\delta_{ik}=0\ (i=1,\ldots,r;\ k=i+1,\ldots,m)$. Then (51) for $i=j=r+1$ becomes

$$\sum_{k=r+2}^{m}\delta_{r+1k}A_{kr+1}=0,$$

whence $\delta_{r+1k}=0\ (k=r+2,\ldots,m)$. Hence (δ_{ij}) is of the form (50). For $j\overline{\gtrless} i$, conditions (51) are now identities; for $j\overline{\lessgtr} i-1$, (51) becomes

$$\sum_{k=j+1}^{i}\delta_{ik}A_{kj}=\sum_{l=j}^{i-1}\alpha_{il}\delta_{lj}\qquad (j\overline{\lessgtr} i-1).\qquad (52)$$

Since $\delta_{jj}\neq 0$, (52) serves to express α_{ij} in terms of $\alpha_{il}\,(l=j+1,\ldots,i-1)$, A's and δ's and hence ultimately in terms of the A's and δ's.

23. The $p^{\mu n}(p^n-1)^{m-1}(p^{2n}-1)$ *operators transforming* $\{\alpha_{ss-1}=0\}$ *into itself are*

$$(\delta_{ij}),\ \delta_{ij}=0\,(j>i)\ \textit{except}\ \delta_{s-1s},\ \delta_{ii}\neq 0\,(i=1,\ldots,m;\ i\neq s,s-1),\qquad (53)$$
$$\Delta_{s-1s}\equiv\delta_{s-1s-1}\delta_{ss}-\delta_{s-1s}\delta_{ss-1}\neq 0.$$

For $i=j=1,2,\ldots,s-2$ in (51), we get $\delta_{ij}=0\ (i=1,\ldots,s-2;\ j=i+1,\ldots,m)$ as in §22. For $i=j=s-1$, (51) becomes $\sum\limits_{k=s+1}^{m}\delta_{s-1k}A_{ks-1}=0$, whence $\delta_{s-1k}=0$, $k\overline{\gtrless} s+1$. For $i=j=s$, (51) gives $\delta_{sk}=0$, $k\overline{\gtrless} s+1$. By induction we prove that $\delta_{ij}=0, j>i, i\overline{\gtrless} s$. Hence (δ_{ij}) is of the form (53). Conditions (51) are now identities for $j\overline{\gtrless} i$. Let next $j\overline{\lessgtr} i-1$. For $j\neq s,\ s-1$, (51) serves to express α_{ij} in terms of the $\alpha_{il}\,(l=j+1,\ldots,i-1)$, A's and δ's, since the coefficient of α_{ij} is $\delta_{jj}\neq 0$, while that of $\alpha_{il}\,(l<j)$ is zero. The two conditions (51) given by $j=s$, $j=s-1$, in which we may assume that $i>s$, serve to express α_{is-1} and α_{is} in terms of the $\alpha_{il}\,(l=s+1,\ldots,i-1)$, since $\Delta_{s=1s}\neq 0$.

24. *If the non-vanishing* c_i *are* $c_{i_1},\ldots,c_{i_v}$, $v\overline{\gtrless} 2$, the $p^{\mu n}\,(p^n-1)^{m-v+1}$ *operators transforming into itself* $\left\{\sum\limits_{i=2}^{m}c_i\alpha_{ii-1}=0\right\}$ *are*

$$(\delta_{ij}),\ \delta_{ij}=0\,(j>i),\ \delta_{ii}\neq 0,\ \delta_{ii}\delta_{i-1i-1}^{-1}\,(i=i_1,\ldots,i_v)\ \textit{all equal}.\qquad (54)$$

Every $A_{ij}\,(j<i-1)$ and any particular A_{ii-1} may be taken arbitrary. Hence as in §22, every $\delta_{ij}=0\,(j>i)$, while the remaining conditions reduce to (52) and determine the α_{ij} in terms of the A's and δ's. In particular, for $j=i-1$, (52) gives

$$\delta_{ii}A_{ii-1}=\alpha_{ii-1}\delta_{i-1i-1} \qquad (i=2,\ldots,m).$$

Hence

$$\sum_{i=2}^{m}c_i\alpha_{ii-1}=\sum_{i=2}^{m}c_iA_{ii-1}\delta_{ii}\delta_{i-1i-1}^{-1}=0$$

must follow from $\sum_{i=2}^{m}c_iA_{ii-1}=0$, giving the final conditions (54).

25. If every $D_{ij}=0\,(i\neq s, j\neq s)$, group (25) takes the form

$$\left\{\alpha_{ss-1}=0,\sum_{\substack{i=2,\ldots,m\\ i\neq s}}\gamma_i\alpha_{ii-1}=0\right\}, \qquad (\text{not every } \gamma_i=0), \qquad (55)$$

and will be considered in §§26, 27. Suppose here that for each integer s, $2\overline{\gtrless}s\overline{\gtrless}m$, a certain $D_{ij}\neq 0\,(i\neq s, j\neq s)$. Then for any given s we may take α_{ss-1} arbitrary and determine the remaining α_{ii-1} to satisfy the conditions on (25). Hence, as in §22, an operator (δ_{ij}) commutative with (25) has every $\delta_{ij}=0\ (j>i)$. As in §24, the equations

$$\sum_{i=2}^{m}c_iA_{ii-1}\delta_{ii}\delta_{i-1i-1}^{-1}=0,\sum_{i=2}^{m}k_iA_{ii-1}\delta_{ii}\delta_{i-1i-1}^{-1}=0$$

must follow from

$$\sum_{i=2}^{m}c_iA_{ii-1}=0,\sum_{i=2}^{m}k_iA_{ii-1}=0.$$

Let $D_{rt}\neq 0$, so that $A_{ii-1}(i=2,\ldots,m; i\neq r,t)$ may be given arbitrary values. Equating the values of A_{rr-1} and A_{tt-1} obtained by solving the two pairs of equations, we get

$$\frac{\delta_{r-1r-1}}{\delta_{rr}}\begin{vmatrix} c_i\delta_{ii}\delta_{i-1i-1}^{-1} & c_t\\ k_i\delta_{ii}\delta_{i-1i-1}^{-1} & k_t\end{vmatrix}=\begin{vmatrix}c_ic_t\\ k_ik_t\end{vmatrix}, \quad \frac{\delta_{t-1t-1}}{\delta_{tt}}\begin{vmatrix} c_r & c_i\delta_{ii}\delta_{i-1i-1}^{-1}\\ k_r & k_i\delta_{ii}\delta_{i-1i-1}^{-1}\end{vmatrix}=\begin{vmatrix}c_rc_i\\ k_rk_i\end{vmatrix}$$

for $i=2,\ldots,m; i\neq r,t$. Hence must

$$D_{it}(\delta_{rr}\delta_{i-1i-1}-\delta_{ii}\delta_{r-1r-1})=0,\ D_{ir}(\delta_{tt}\delta_{i-1i-1}-\delta_{ii}\delta_{t-1t-1})=0.$$

If $D_{it}=D_{ir}=0$, then $c_i=k_i=0$ and the preceding are identities.

THEOREM. *For each integer* s, $2 \overline{\gtrless} s \overline{\gtrless} m$, *let at least one of the* $D_{ij} \neq 0$ $(i \neq s, j \neq s)$. *Let, in particular,* $D_{rt} \neq 0$. *Denote by* $i_1, \ldots, i_v$ *the integers* i *such that* $2 \overline{\gtrless} i \overline{\gtrless} m$, $i \neq r$, $i \neq t$, *and such that* c_i, k_i *are not both zero. If there occurs among the* $i_1, \ldots i_v$ *an integer* i *for which* $D_{it} \neq 0$, $D_{ir} \neq 0$, *then the* $p^{\mu n}(p^n - 1)^{m-v-1}$ *operators transforming* (25) *into itself are the* (δ_{ij}),

$$\delta_{ij} = 0\,(j > i), \; \frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{rr}}{\delta_{r-1r-1}} = \frac{\delta_{tt}}{\delta_{t-1t-1}} \;(i = i_1, \ldots, i_v). \tag{56}$$

In the contrary case, let $i_1, \ldots, i_w$ *be the integers* i *for which* $D_{it} \neq 0$, $D_{ir} = 0$, *and* $i_{w+1}, \ldots, i_v$ *the remaining* i *for which therefore* $D_{it} = 0$, $D_{ir} \neq 0$; *then the* $p^{\mu n}(p^n - 1)^{m-v}$ *operators transforming* (25) *into itself are the* (δ_{ij}),

$$\delta_{ij} = 0\,(j > i), \; \frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{rr}}{\delta_{r-1r-1}} \;(i = i_1, \ldots, i_w),$$
$$\frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{tt}}{\delta_{t-1t-1}} \;(i = i_{w+1}, \ldots, i_v). \tag{57}$$

26. Consider group (55) with at least two γ_i not zero. Any particular A_{ii-1} $(i \neq s)$ can be taken arbitrarily. As in §23, every $\delta_{ij} = 0\,(j > i)$ except possibly δ_{s-1s}, while those conditions (51) which do not now reduce to identities serve to express the α_{ij} $(j \overline{\gtrless} i - 1)$ in terms of the A's and δ's. In particular, for $j = i - 1$, $i \neq s$, $i \neq s + 1$, (51) gives

$$\alpha_{ii-1} = A_{ii-1}\delta_{ii}\delta_{i-1i-1}^{-1}\,(i \neq s-1, s, s+1),$$
$$\alpha_{s-1s-2} = \delta_{s-2s-2}^{-1}\,(\delta_{s-1s-1}A_{s-1s-2} + \delta_{s-1s}A_{ss-2});$$

while for $i = s + 1$, $j = s - 1$, and for $i = s + 1$, $j = s$, (51) gives

$$\delta_{s+1s+1}A_{s+1j} = \alpha_{s+1s}\delta_{sj} + \alpha_{s+1s-1}\delta_{s-1j}\,(j = s-1, s),$$

which determine $\alpha_{s+1s}, \alpha_{s+1s-1}$. The equation obtained upon substituting these values of the α_{ii-1} in $\Sigma\gamma_i\alpha_{ii-1} = 0$ must follow from $\Sigma\gamma_i A_{ii-1} = 0$. The coefficients of A_{ss-2} and A_{s+1s-1} must vanish, whence

$$\gamma_{s-1}\delta_{s-1s} = 0, \; \gamma_{s+1}\delta_{s-1s} = 0.$$

Further, for the non-vanishing γ_i, there must result equal values of

$$\gamma_i\delta_{ii}\delta_{i-1i-1}^{-1} : \gamma_i, \quad \gamma_{s+1}\delta_{s+1s+1}\delta_{s-1s-1}\Delta_{s-1s}^{-1} : \gamma_{s+1}\,(i \neq s, s+1).$$

THEOREM. *Let the non-vanishing* γ_i *be* $\gamma_{i_1}, \ldots, \gamma_{i_v}$, $v \overline{>} 2$. *If either* $\gamma_{s+1} \neq 0$ *or* $\gamma_{s-1} \neq 0$, *the* $p^{\mu n}(p^n-1)^{m-v+1}$ *operators transforming* (55) *into itself are given by* (54). *If* $\gamma_{s+1} = \gamma_{s-1} = 0$, *the* $p^{\mu n}(p^n-1)^{m-v}(p^{2n}-1)$ *operators are given by* (53), *subject to the further condition that the* $\delta_{ii}\delta_{i-1i-1}^{-1}$ $(i = i_1, \ldots, i_v)$ *shall be equal.*

27. Consider group (55) with $\gamma_i = 0\ (i \neq r)$, viz. $\{\alpha_{ss-1} = 0,\ \alpha_{rr-1} = 0\}$. We may take $r > s$. As in §23,

$$\delta_{ij} = 0\ (i = 1, \ldots, s-2;\ j > i),\ \delta_{s-1k} = 0\ (k = \overline{>} s+1),\ \sum_{k=s+1}^{m} \delta_{sk} A_{ks} = 0.$$

If $r = s+1$, the latter gives merely $\delta_{sk} = 0\ (k \overline{>} s+2)$. Then (51) for $i = j = s+1$ gives $\delta_{s+1k} = 0\ (k \overline{>} s+2)$. By induction, $\delta_{ik} = 0$, $(i = s+1, \ldots, m;\ k > i)$. Hence every $\delta_{ij} = 0\ (j > i)$ except δ_{s-1s} and $\delta_{ss+1} \equiv \delta_{r-1r}$. Similarly, if $r > s+1$, we get $\delta_{ij} = 0\ (j > i)$ except δ_{s-1s}, δ_{r-1r}.

Conditions (51) are now identities if $j \overline{<} i$. Let next $j \overline{<} i-1$.

Let first $r > s+1$. Then $\delta_{ii} \neq 0\ (i \neq s-1, s, r-1, r)$, $\Delta_{s-1s} \neq 0$, $\Delta_{r-1r} \neq 0$. For $j \neq s-1, s, r-1, r$, (51) serves to express α_{ij} in terms of the $\alpha_{il}\ (l = j+1, \ldots, i-1)$, A's and δ's. Proceeding as at the end of §23, we conclude that conditions (51) for $j \overline{<} i-1$ merely serve to express the α_{ij} in terms of the A's and δ's.

For $r = s+1$, we have $\delta_{ii} \neq 0\ (i \neq s-1, s, s+1)$, and

$$\Delta \equiv \begin{vmatrix} \delta_{s-1s-1} & \delta_{s-1s} & 0 \\ \delta_{ss-1} & \delta_{ss} & \delta_{ss+1} \\ \delta_{s+1s-1} & \delta_{s+1s} & \delta_{s+1s+1} \end{vmatrix} \neq 0.$$

For $j \neq s-1, s+1$, (51) serves to express α_{ij} in terms of the $\alpha_{il}\ (l = j+1, \ldots, i-1)$, A's, and δ's. For $j = s-1, s, s+1$, with $i > s+1$, (51) determines α_{is-1}, α_{is}, α_{is+1}, the determinant of their coefficients being Δ. There remain the cases $i = s+1, j = s$; $i = s+1, j = s-1$; $i = s, j = s-1$, for which (51) becomes, respectively,

$$0 = \alpha_{s+1s-1}\delta_{s-1s},\ \delta_{s+1s+1}A_{s+1s-1} = \alpha_{s+1s-1}\delta_{s-1s-1},\ \delta_{ss+1}A_{s+1s-1} = 0.$$

Hence $\delta_{s-1s} = \delta_{ss+1} = 0$, whence every $\delta_{ii} \neq 0$. The second condition thus determines α_{s+1s-1}. Hence the $\alpha_{ij}\ (j \overline{<} i-1)$ are determined in terms of the A's, δ's.

THEOREM. *The* $p^{\mu n}(p^n-1)^m$ *operators transforming* $\{\alpha_{ss-1}=\alpha_{s+1s}=0\}$ *into itself are given by* (50). *If* $r>s+1$, *the* $p^{\mu n}(p^n-1)^{m-2}(p^{2n}-1)^2$ *operators transforming* $\{\alpha_{ss-1}=0,\ \alpha_{rr-1}=0\}$ *into itself are the* (δ_{ij}),

$$\delta_{ij}=0\ (j>i)\ \textit{except}\ \delta_{s-1s}\ \textit{and}\ \delta_{r-1r},\ \Delta_{s-1s}\neq 0,\qquad \Delta_{r-1r}\neq 0,\ \delta_{ii}\neq 0\ (i\neq s-1,\ s,\ r-1,\ r). \tag{58}$$

28. For the group (28), we proceed as in §23 except for the step $i=j=s-1$, instead of which we employ (51) for $i=s-1,\ j=s$, and get $\delta_{s-1k}=0\ (k\overline{>}s+1)$. The values of the α_{ii-1} and α_{s+1s-1} are the same as in §26. Substituting them in $\alpha_{s+1s-1}=\alpha_{s-1s-2}+\Sigma x_i\alpha_{ii-1}$, replacing A_{s+1s-1} by $A_{s-1s-2}+\Sigma x_iA_{ii-1}$, and equating the coefficients of each A_{ii-1} and of A_{ss-2}, we get

$$\delta_{s-1s}=\delta_{ss-1}=0,\ x_i\delta_{s+1s+1}\delta_{ss}\Delta_{s-1s}^{-1}=x_i\delta_{ii}\delta_{i-1i-1}^{-1}(i\neq s,\ s+1).$$

THEOREM. *If the non-vanishing* x_i *are* $x_{i_1},\ldots,x_{i_v}$, *the* $p^{n(\mu-1)}(p^n-1)^{m-v}$ *operators transforming* (28) *into itself are the* (δ_{ij}),

$$\delta_{ij}=0\,(j>i),\ \delta_{ss-1}=0,\ \delta_{ii}\delta_{i-1i-1}^{-1}=\delta_{s+1s+1}\delta_{s-1s-1}^{-1}\ (i=i_1,\ldots,i_v). \tag{59}$$

29. Consider group (21) with $k_1=1,\ k_{s+1}=0$ (§13, end). We can take every A arbitrary except A_{ss-1} and A_{s+1s-1}. By (51) for $i=j=1,\ldots,s-2$, we get $\delta_{ij}=0\ (i=1,\ldots,s-2;j>i)$. For $i=s-1,j=s$, (51) gives $\delta_{s-1k}=0\ (k\overline{>}s+1)$. Then for $i=j=s-1$, (51) becomes $\delta_{s-1s}A_{ss-1}=0$. But we may take $A_{ss-1}\neq 0$. Hence $\delta_{s-1k}=0\ (k\overline{>}s)$. Then (51) for $i=j=s,\ldots,m$ gives $\delta_{ik}=0\ (i\overline{>}s,k>i)$. Now (52), for $j=i-1,\ j=i-2$, give

$$\alpha_{ii-1}=\frac{A_{ii-1}\delta_{ii}}{\delta_{i-1i-1}},\ \alpha_{ii-2}=\frac{\delta_{ii-1}A_{i-1i-2}+\delta_{ii}A_{ii-2}}{\delta_{i-2i-2}}-\frac{A_{ii-1}\delta_{ii}\delta_{i-1i-2}}{\delta_{i-1i-1}\delta_{i-2i-2}}. \tag{60}$$

Substituting these in the two equations (21), eliminating A_{s+1s-1} and then A_{s+1s} by (21) written in the A's, we get

$$\frac{\delta_{ss}}{\delta_{s-1s-1}}=\frac{\delta_{s+1s+1}}{\delta_{ss}},\ \sum_{\substack{i=2,\ldots,m\\ i\neq s,s+1}}k_iA_{ii-1}\Big(\frac{\delta_{ii}}{\delta_{i-1i-1}}-\frac{\delta_{s+1s+1}}{\delta_{s-1s-1}}\Big)-\frac{A_{ss-1}\beta}{\delta_{ss}\delta_{s-1s-1}}=0,$$

where β is given below. This must be an identity in the A's.

THEOREM. *Denote by* Γ *the group* (21) *with* $k_1 = 1, k_{s+1} = 0$. *Let the non-vanishing* k_i *be* $k_{i_1}, \ldots, k_{i_v}$. *The* $p^{n(\mu-1)}(p^n - 1)^{m-v-1}$ *operators commutative with* Γ *are the* (δ_{ij}),

$$\delta_{ij} = 0\ (j > i),\ \frac{\delta_{ss}}{\delta_{s-1s-1}} = \frac{\delta_{s+1s+1}}{\delta_{ss}},\ \frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{s+1s+1}}{\delta_{s-1s-1}}\ (i = i_1, \ldots, i_v),$$
$$\beta \equiv c_s\delta_{ss-1}\delta_{s+1s+1} + c_{s+1}\delta_{s+1s}\delta_{ss} = 0. \tag{61}$$

30. THEOREM. *Denote by* H *the group* (23) *with* $k_1 = 1$, $k_{s+1} = 0$. *Let the non-vanishing* k_i *be* $k_{i_1}, \ldots, k_{i_v}$. *The operators commutative with* H *are*

$$(\delta_{ij}),\ \delta_{ij} = 0\ (j > i),\ \frac{\delta_{ii}}{\delta_{i-1i-1}} = \frac{\delta_{s+1s+1}}{\delta_{s-1s-1}} = \frac{\delta_{s+2s+2}}{\delta_{ss}}(i = i_1, \ldots, i_v), \tag{62}$$
$$C \equiv c_s\delta_{ss-1}\delta_{s+1s+1} + c_{s+1}\delta_{s+1s}\delta_{ss} + c_{s+2}\delta_{s+2s+1}\delta_{s-1s-1} = 0,$$

with in case $c_{s+1} \neq 0$, *the further condition* $\delta_{ss}^2 = \delta_{s+1s+1}\delta_{s-1s-1}$.

The proof proceeds as in §29. We substitute the values (60) in the two equations (23), eliminate A_{s+1s-1} and then A_{ss-1} by (23) written in the A's. From $\Sigma c_i\alpha_{ii-1} = 0$ follows, as in §24, that the $\delta_{ii}\delta_{i-1i-1}^{-1}(i = i_1, \ldots, i_v)$ are equal. From $f_\alpha + \Sigma k_i\alpha_{ii-i} = 0$, we get

$$\left(\frac{\delta_{s+1s+1}}{\delta_{s-1s-1}} - \frac{\delta_{s+2s+2}}{\delta_{ss}}\right)\left(c_{s+2}A_{s+2s} - \tfrac{1}{2}c_{s+2}A_{s+2s+1}A_{s+1s} - \frac{c_{s+2}A_{s+2s+1}\delta_{s+1s}}{\delta_{s+1s+1}}\right)$$
$$+ \sum_{\substack{i=2,\ldots,m\\ i\neq s,\, s+1}} k_iA_{ii-1}\left(\frac{\delta_{ii}}{\delta_{i-1i-1}} - \frac{\delta_{s+1s+1}}{\delta_{s-1s-1}}\right) - \frac{CA_{s+1s}}{\delta_{ss}\delta_{s-1s-1}} = 0.$$

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